Stochastic dynamo model for subcritical transition

Sergei Fedotov,[1,*] Irina Bashkirtseva,[2] and Lev Ryashko[2]
[1] *School of Mathematics, The University of Manchester, M60 1QD, United Kingdom*
[2] *Department of Mathematical Physics, Ural State University, Lenin Av., 51, 620083 Ekaterinburg, Russia*
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The effects of stochastic perturbations in a nonlinear $\alpha\Omega$-dynamo model are investigated. By using transformation of variables we identify a "slow" variable that determines the global evolution of the non-normal $\alpha\Omega$-dynamo system in the subcritical case. We apply an adiabatic elimination procedure to derive a closed stochastic differential equation for the slow variable for which the dynamics is determined along one of the eigenvectors of the full system. We derive the corresponding Fokker-Planck equation and show that the generation of a large scale magnetic field can be regarded as a first-order phase transition. We show that the an advantage of the reduced system is that we have explicit expressions for both the stochastic and deterministic potentials. We also obtain the stationary solution of the Fokker-Planck equation and show that an increase in the intensity of the multiplicative noise leads to qualitative changes in the stationary probability density function. The latter can be interpreted as a noise-induced phase transition. By a numerical simulation of the stochastic galactic dynamo model, we show that the qualitative behavior of the "empirical" stationary pdf of the slow variable is accurately predicted by the stationary pdf of the reduced system.

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I. INTRODUCTION

Non-normal transient growth has attracted enormous attention in recent years because of its importance for our understanding of the early stages of turbulence [1,2]. It explains the subcritical transition to turbulence when the laminar flow changes to a turbulent regime without linear instability [3]. The generation of large-scale magnetic fields of stars and galaxies can also be understood in terms of non-normal growth [4,5]. The crucial role of non-normality of the induction equation for a magnetic field has been emphasized in Ref. [6].

Extensive investigations have been devoted to stochastically forced dynamical systems involving a non-normal operator (see, for example, Refs. [7,8] and references therein). The main reason for this is that these systems have an extraordinary sensitivity to random perturbations. As a consequence, there is a large amplification of the variances. Although a great deal of progress has been made in this theory, the results are restricted to a linear problem of the derivation of equations for second moments. Our understanding of nonlinear stochastic systems with non-normality is much less complete [10,11]. Numerical solutions for a two-variables model have been obtained in Ref. [12]. It seems natural to extend these results and study the global behavior of a non-normal dynamical system analytically.

In Ref. [13] we found the criteria for the stochastic amplification of the magnetic field during the kinematic (linear) stage only. The purpose of this paper is to analyze the *nonlinear* $\alpha\Omega$ dynamo. We are concerned with a stochastic non-normal dynamical system near bifurcation point when only one "slow" variable is required to determine the global evolution of a system with two variables. We intend to consider a subcritical case and derive a closed stochastic differential equation for the slow variable by using a transformation of variables and a technique of the adiabatic elimination [14–16]. The aims are (i) to describe the magnetic field growth as a stochastic process with multiple stationary states (a first-order phase transition) and (ii) to show that an increase in the intensity of the multiplicative noise might lead to qualitative changes in the stationary probability density function of the slow variable.

II. STATEMENT OF THE PROBLEM

We consider the nonlinear stochastic $\alpha\Omega$-dynamo model [11]

$$\frac{dB_r}{dt} = -\delta\varphi_\alpha(B_r,B_\varphi)B_\varphi - \varepsilon\varphi_\beta(B_r,B_\varphi)B_r + \sqrt{2\sigma_r(B_r,B_\varphi)}\frac{dW_r}{dt},$$

$$\frac{dB\varphi}{dt} = -gB_r - \varepsilon\varphi_\beta(B_r,B_\varphi)B_\varphi + \sqrt{2\sigma_\varphi(B_r,B_\varphi)}\frac{dW_\varphi}{dt}, \quad (1)$$

where B_r and B_φ are the radial and azimuthal components of the magnetic field and $W_r(t)$ and $W_\varphi(t)$ are the uncorrelated standard Wiener processes. The nonlinear functions $\varphi_\alpha(B_r,B_\varphi)$ and $\varphi_\beta(B_r,B_\varphi)$ describe the dynamo quenching [9,11] and have the following properties: $\varphi_\alpha(0,0) = \varphi_\beta(0,0) = 1$. The functions $\sigma_r(B_r,B_\varphi)$ and $\sigma_r(B_r,B_\varphi)$ are the noise intensity parameters.

We assume that the system (1) has symmetric pairs of fixed points, in addition to the origin $(0,0)$ which is always linearly stable. Since for the $\alpha\Omega$ dynamo, the differential rotation dominates over the α effect, the parameters ε are δ are small, while $g\sim 1$. This makes the linearized operator, namely, the matrix

*Author to whom correspondence should be addressed.

$$\begin{bmatrix} -\varepsilon & -\delta \\ -g & -\varepsilon \end{bmatrix} \tag{2}$$

highly non-normal [3]. Recall that in general an operator is said to be non-normal if it does not commute with its adjoint in the corresponding scalar product. This model involves a non-normal stable linear part leading to transient growth. We assume that for the system (1) without random terms only small but finite initial perturbations can escape from the basin of attraction of the fixed point at the origin. The presence of noise in the right hand side of Eq. (1) may lead to a transition, even for zero initial conditions: $B_r(0)=0$, $B_\varphi(0)=0$. This is important since in a practical situation random perturbations may often be what induce the subcritical magnetic field generation.

Let us first consider the deterministic dynamical system (1) in a linear case. We can expand the RHS of Eq. (1) in a Taylor series near equilibrium at $B_r=0, B_\varphi=0$ to obtain

$$\begin{bmatrix} \dfrac{dB_r}{dt} \\ \dfrac{dB_\varphi}{dt} \end{bmatrix} = \begin{bmatrix} -\varepsilon & -\delta \\ -g & -\varepsilon \end{bmatrix} \begin{bmatrix} B_r \\ B_\varphi \end{bmatrix}. \tag{3}$$

The matrix of this system has two eigenvalues $\lambda_1=-\varepsilon+\sqrt{g\delta}, \lambda_2=-\varepsilon-\sqrt{g\delta}$ and two corresponding eigenvectors

$$h_1 = \begin{bmatrix} -\mu \\ 1 \end{bmatrix}, \quad h_2 = \begin{bmatrix} \mu \\ 1 \end{bmatrix}, \quad \mu = \sqrt{\frac{\delta}{g}} \ll 1. \tag{4}$$

The characteristic feature of these linearized equations is that for small μ eigenvectors h_1 and h_2 are almost parallel. This means that the linear system (3) is highly non-normal. In the *subcritical* case ($\varepsilon > \sqrt{g\delta}$) although both eigenvalues, λ_1 and λ_2, are negative, non-normality may lead to a large transient growth of $B_r(t)$ prior to an eventual exponential decay [3,12].

III. TRANSFORMATION OF VARIABLES

Let us consider the dynamics of the system (1) along eigenvector h_1 in Eq. (4). Let us introduce scalar variables $u(t)$ and $v(t)$ by using the eigenvectors as a basis [16]

$$\begin{bmatrix} B_r(t) \\ B_\varphi(t) \end{bmatrix} = \begin{bmatrix} -\mu & \mu \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \tag{5}$$

[or $B_r=\mu(v-u)$, $B_\varphi=v+u$]. Under the change of variables [Eq. (5)], the nonlinear stochastic system (1) becomes

$$\frac{du}{dt} = \frac{1}{2\mu}[(m_\alpha - g\mu^2)v + (g\mu^2 - 2m_\beta\mu + m_\alpha)u] + \frac{\gamma_\varphi^{1/2}}{2}\frac{dW_\varphi}{dt} - \frac{\gamma_r^{1/2}}{2\mu}\frac{dW_r}{dt},$$

$$\frac{dv}{dt} = \frac{1}{2\mu}[(-2m_\beta\mu - m_\alpha - g\mu^2)v + (g\mu^2 - m_\alpha)u] + \frac{\gamma_r^{1/2}}{2\mu}\frac{dW_r}{dt} + \frac{\gamma_\varphi^{1/2}}{2}\frac{dW_\varphi}{dt}, \tag{6}$$

where

$$m_\alpha = \delta\varphi_\alpha(\mu(v-u), v+u), \quad m_\beta = \varepsilon\varphi_\beta(\mu(v-u), v+u),$$
$$\gamma_r = 2\sigma_r(\mu(v-u), v+u), \quad \gamma_\varphi = 2\sigma_\varphi(\mu(v-u), v+u). \tag{7}$$

In the linear approximation, the system (6) can be rewritten in a simple diagonal form

$$\frac{du}{dt} = \lambda_1 u, \quad \frac{dv}{dt} = \lambda_2 v. \tag{8}$$

Under the condition $\varepsilon - \sqrt{g\delta} \ll 1$ when $|\lambda_1| \ll |\lambda_2|$, the variable u can be regarded as the slow variable and v as the "fast" one. We can see from Eq. (6) that the stochastic term involving $\frac{dW_r}{dt}$ is proportional to μ^{-1} ($\mu \ll 1$). This explains the sensitivity of the nonlinear dynamical systems with a non-normal transient growth to random perturbations. Since parameter μ is small, the stochastic terms proportional to $\frac{dW_\varphi}{dt}$ can be neglected compared to $\frac{dW_r}{dt}$. This is a very important result for further analysis since the "weak" noise might induce phase transitions in Eq. (6).

Now we are in a position to use the adiabatic elimination procedure to derive a stochastic equation governing the slow evolution of the variable u [14,17]. The projection of the nonlinear system (6) onto the u axis can be obtained in two steps. First, we set $v=0$ in all nonlinear functions. Second, by setting $dv/dt=0$ we find from Eq. (6) that the "fast" variable $v(t)$ follows the values of the slow variable $u(t)$ as

$$v = \frac{g\mu^2 - m_\alpha^0}{2m_\beta^0\mu + m_\alpha^0 + g\mu^2}u + \frac{(\gamma_r^0)^{1/2}}{2m_\beta^0\mu + m_\alpha^0 + g\mu^2}\frac{dW_r}{dt}, \tag{9}$$

where

$$m_\alpha^0 = m_\alpha(u,0), \quad m_\beta^0 = m_\beta(u,0), \quad \gamma_r^0 = \gamma_r(u,0).$$

This corresponds to the so-called "noisy" adiabatic elimination when the fast variable makes a contribution to the noise term in the v equation [14]. Substitution of Eq. (9) into Eq. (6) gives

$$\frac{du}{dt} = b(u) + \sqrt{2\sigma(u)}\frac{dW}{dt}, \tag{10}$$

where

$$b(u) = \frac{2\mu u(gm_\alpha^0 - (m_\beta^0)^2)}{2m_\beta^0\mu + m_\alpha^0 + g\mu^2},$$
$$\sigma(u) = \frac{1}{2}\left(\frac{g\mu + m_\beta^0}{2m_\beta^0\mu + m_\alpha^0 + g\mu^2}\right)^2\gamma_r^0, \tag{11}$$

and $W(t)$ is the standard Wiener processes. Equation (10) without the stochastic term, namely, $du/dt=b(u)$, explains the subcritical transition. There exists a threshold value of u, above which the solution grows $[b(u)>0]$ and below which it decays $[b(u)<0]$. The advantage of the new equation (10) is that one can introduce the deterministic potential

$$U(u) = -\int_0^u b(z)dz. \tag{12}$$

So, Eq. (10) can be rewritten in the form of a random walk in a potential field U,

$$\frac{du}{dt} = -\frac{\partial U}{\partial u} + \sqrt{2\sigma(u)}\frac{dW}{dt}. \tag{13}$$

It should be noted that one cannot identify the deterministic potential for the dynamical system (1).

IV. STOCHASTIC POTENTIAL AND PROBABILITY DENSITY FUNCTION FOR THE SLOW VARIABLE

The Fokker-Planck equation for the probability density function (pdf) $p(t,u)$ corresponding to the Stratonovich stochastic differential equations (SDE) (10) is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial u}\left[\left(b(u) + \frac{1}{2}\sigma'(u)\right)p\right] + \frac{\partial^2}{\partial u^2}[\sigma(u)p]. \tag{14}$$

The stationary solution to this equation can be written in terms of the stochastic potential $V(u)$,

$$p_{st}(u) = N\exp\left[-\frac{V(u)}{\sigma(u)}\right],$$
$$V(u) = -\sigma(u)\left(\int_0^u \frac{b(z)}{\sigma(z)}dz - \frac{1}{2}\ln\sigma(u)\right). \tag{15}$$

The stationary points of $p_{st}(u)$ can be found from the equation $dp_{st}/du=0$ or

$$b(u) - \frac{1}{2}\sigma'(u) = 0. \tag{16}$$

It is clear that this equation is different from the equation $b(u)=0$ determining the critical points of deterministic dynamical system $du/dt=b(u)$. It is also clear that $V(u)$ is not equal to the deterministic potential $U(u)$ (12). The stationary states of the deterministic system occur at the extrema of $U(u)$. It turns out that both the stochastic, $V(u)$, and deterministic, $U(u)$, potentials are especially suited to analyze the processes of subcritical instability.

Rather than attempting to investigate Eq.(10) for all possible dynamo quenching functions [9], we consider the global dynamics for the stochastic dynamo problem in the particular case when [11]

$$m_\alpha^0(u) = \frac{\delta}{1+k_\alpha u^2}, \quad m_\beta^0(u) = \frac{\varepsilon(1+u^2)}{1+(k_\beta+1)u^2}. \tag{17}$$

This corresponds to the nonlinear functions of the form [9]

$$\varphi_\alpha(B_r,B_\varphi) = \frac{1}{1+k_\alpha B_\varphi^2}, \quad \varphi_\beta(B_r,B_\varphi) = \frac{(1+B_\varphi^2)}{1+(k_\beta+1)B_\varphi^2}, \tag{18}$$

where k_α and k_β are constants of order one.

Let us derive now an explicit expression for the "noisy intensity" function $\sigma(u)$ in Eq. (10). We consider both multiplicative noise, dW_1/dt, and additive noise, dW_2/dt, for the dynamo problem involving the functions (18). Recall that the source of multiplicative noise is the random fluctuations of the parameter δ (random α effect) [13]

$$\delta \to \delta + \sqrt{2\sigma_1}dW_1/dt. \tag{19}$$

Thus, the first equation in Eq. (1) can be written as follows:

$$\frac{dB_r}{dt} = -\frac{\delta B_\varphi}{1+k_\alpha B_\varphi^2} - \frac{\varepsilon(1+B_\varphi^2)B_r}{1+(k_\beta+1)B_\varphi^2} + \frac{\sqrt{2\sigma_1}y}{1+k_\alpha y^2}\frac{dW_1}{dt} + \sqrt{2\sigma_2}\frac{dW_2}{dt}, \tag{20}$$

where σ_1 is the multiplicative noise intensity and σ_2 is the additive noise intensity [13]. If we combine the two independent noises dW_1/dt and dW_2/dt into one, then Eq. (20) can be rewritten in the form of the first equation of the system (1),

$$\frac{dB_r}{dt} = -\frac{\delta B_\varphi}{1+k_\alpha B_\varphi^2} - \frac{\varepsilon(1+B_\varphi^2)B_r}{1+(k_\beta+1)B_\varphi^2} + \sqrt{2\sigma_r(B_\varphi)}\frac{dW_r}{dt}, \tag{21}$$

where we have introduced the noise intensity $\sigma_r(B_\varphi)$,

$$\sigma_r(B_\varphi) = \sigma_1\frac{B_\varphi^2}{(1+k_\alpha B_\varphi^2)^2} + \sigma_2. \tag{22}$$

One can see that this noise intensity is independent of B_r. The function $\sigma(u)$ in Eq. (10) can be found as follows. Substitution of Eq. (22) into Eq. (7) and taking $v=0$ give us $\gamma_r^0=\gamma_r(u,0)$. From Eq. (11) one can get

$$\sigma(u) = \left(\frac{g\mu + m_\beta^0}{2m_\beta^0(u)\mu + m_\alpha^0(u) + g\mu^2}\right)^2\left(\sigma_1\frac{u^2}{(1+k_\alpha u^2)^2} + \sigma_2\right). \tag{23}$$

Now we are in a position to discuss the effects of stochastic perturbations on a non-normal dynamical system near the bifurcation point in the subcritical case. First, let us consider equation (10) without random perturbations. By using Eqs. (11) and (17), one can find the equation

$$\frac{\varepsilon^2}{g\delta}(1+u^2)^2(1+k_\alpha u^2) - [1+(k_\beta+1)u^2]^2 = 0$$

determining nontrivial stationary points for the deterministic equation $du/dt=b(u)$. If we take $\varepsilon=0.1$, $\delta=0.01$, $k_\alpha=k_\beta=1$, then for $g<0.84$ there exists only one stable equilibrium point $u=0$. For $0.84<g<1$ there are three stable and two unstable points. For $g>1$ there are two stable nonzero equilibrium points, and $u=0$ becomes unstable. For $\varepsilon=0.1, \delta$

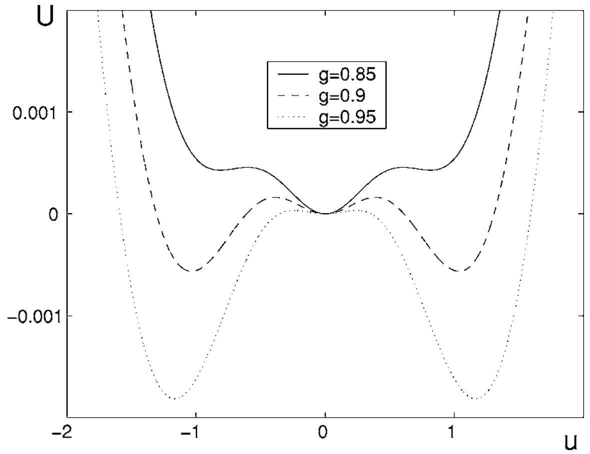

FIG. 1. Deterministic potentials $U(u)$ of u system (13) for the different values g.

$=0.01$, the bifurcation value $g=1$ separates subcritical ($g<1$) and supercritical ($g>1$) zones. In this paper we consider only the subcritical case: $0.84<g<1$ for which $|\lambda_1| \ll |\lambda_2|$ (the criteria for the adiabatic elimination procedure).

Figure 1 shows how the deterministic potential $U(u)$ changes as the parameter g increases. The metastable behavior arises when $U(u)$ has three minima in the subcritical zone as g varies from 0.84 to 1. Recall that a metastable state is defined as a state at which the potential $U(u)$ has a local minimum but not an absolute minimum. It is well known [14] that for additive noise with the intensity σ the transition time, T, from a metastable state, u_1, to a stable one, u_3, depends strongly on the barrier height $\Delta U = U(u_2) - U(u_1)$, where u_2 is the local maximum. One can estimate that $T \sim \exp(\Delta U/\sigma)$. Figure 1 shows that the barrier height, separating the two minima, decreases as g increases up to 1.Of course, the metastable state at $u_1=0$ ceases to exist when $g=1$. Let us mention that one can get an analytic expression for the transition time T in the general case involving both multiplicative and additive noises [14]. This is an advantage of the u system (10) with an explicit expression for the potential $U(u) = -\int^u b(z)dz$ over the dynamical system with two variables (1). It follows from the above discussion that in the subcritical case, the generation of a large-scale magnetic field can be regarded as a first-order phase transition.

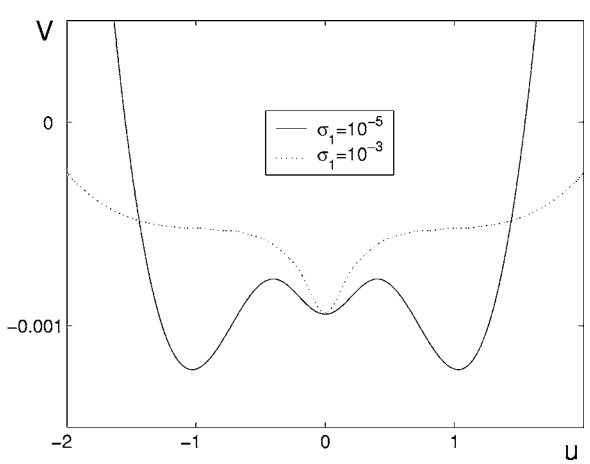

FIG. 2. Stochastic potentials $V(u)$ of u system (13) for the different values of the multiplicative noise intensities σ_1.

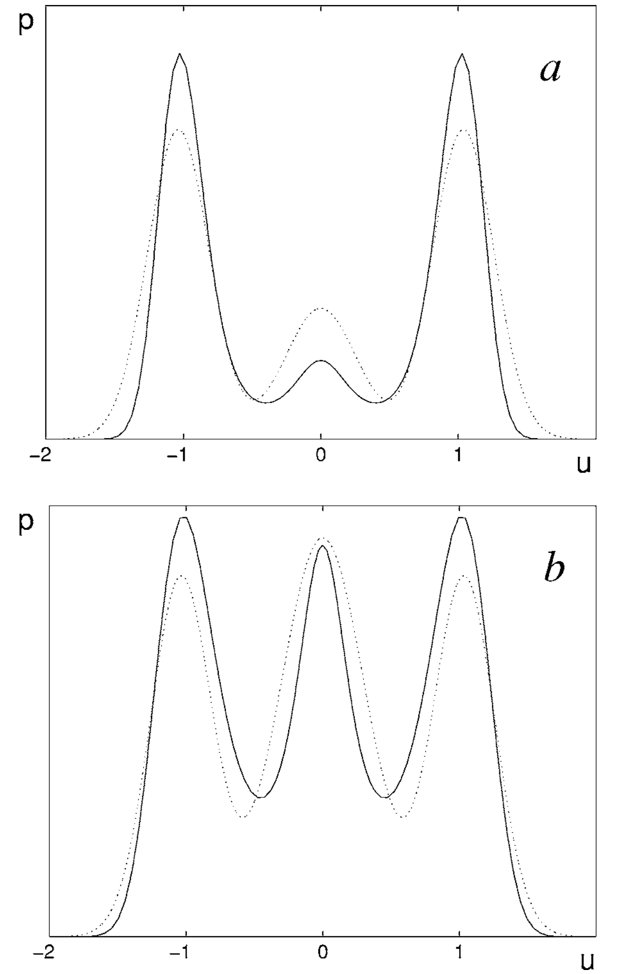

FIG. 3. Probability density functions of the u system (solid) and empiric probability density functions of the u-v-system (dotted) for different values of the intensity of multiplicative noise σ_1: (a) $\sigma_1=0$, (b) $\sigma_1=3\times10^{-5}$ and fixed values $g=0.9$, $\sigma_2=10^{-5}$.

Figure 2 shows the stochastic potential $V(u)$ for different values of the multiplicative noise intensity σ_1 ($g=0.9$, $\sigma_2=10^{-5}$). It can be seen that for $\sigma_1=10^{-5}$, the potential $V(u)$ has the same shape as $U(u)$, such that the stationary state $u_1=0$ can be regarded as a metastable state (local minimum). Increasing parameter σ_1 leads to a qualitative change of the stochastic potential: the metastable state $u_1=0$ becomes a stable one ($\sigma_1=10^{-3}$). This implies that the increase of the multiplicative noise intensity σ_1 induces a stochastic stabilization of the dynamical system at $u_1=0$ in the long-time limit. The latter can be interpreted as a noise-induced phase transition [15].

By a numerical simulation of the dynamical system with two variables (6) we obtain a stationary probability density function for u, and compare it to the analytical results (15) corresponding to reduced dynamics on the slow manifold along eigenvector h_1 (4). Figure 3 shows that the qualitative behavior of the “empirical” stationary pdf of u corresponding to the u-v-system (6) is accurately predicted by the stationary pdf of the reduced u system.

V. CONCLUSIONS

We have developed a methodology for obtaining a reduced description of the stochastic $\alpha\Omega$-dynamo model near the bifurcation point in the subcritical case. This allows us to study analytically the effects of stochastic perturbations on a non-normal system in terms of reduced dynamics on the slow manifold instead of the full system. By using the transformation of variables, we have identified a slow variable that determines the global evolution of the non-normal $\alpha\Omega$-dynamo system with two variables. We have applied an adiabatic elimination procedure to derive a stochastic differential equation for the slow variable. We have shown that the generation of a large-scale magnetic field can be regarded as a first-order phase transition. We have derived the corresponding Fokker-Planck equation and identify the stochastic and deterministic potentials. We have also obtained the stationary solution of the corresponding Fokker-Planck equation and showed that an increase in the intensity of the multiplicative noise leads to qualitative changes in the stationary probability density function. By a numerical simulation of stochastic differential equations, we have shown that the qualitative behavior of the empirical stationary pdf of the slow variable is accurately predicted by the stationary pdf of the reduced system.

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